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Assumed–Stress Hybrid Elements with Drilling DoF for Nonlinear Analysis of Composite Structures

(NAG–1–1505)

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(NASA-CR-195802) ASSUMED-STRESS HYBRID ELEMENTS WITH DRILLING DoF FOR NONLINEAR ANALYSIS OF COMPOSITE STRUCTURES (Old Dominion Univ.) 53 p

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Grant History

- Initiated on January 10, 1992 at Clemson University as NAG–1–1374; completed no–cost extension on May 15, 1993.

- Initiated on April 16, 1993 at Old Dominion University as NAG–1–1505; no–cost extension until August 15, 1994; renewal proposal submitted.

- Three supplements funded:
  - Suppl. 1 – Modeling and additional element checkout (Carron)
  - Suppl. 2 – Adaptive dynamic relaxation for MPP systems (Oakley)
  - Suppl. 3 – Interface element development (Aminpour)

- Grant personnel
  - N. F. Knight – Principal investigator (Clemson and ODU)
  - G. Rengarajan & V. Deshpande – Clemson GRAs, MS, May 1993
  - D. Oakley – Clemson GRA, PhD, May 1994
  - S. Carron – ODU GRA, MS candidate under NFK
  - M. A. Aminpour – Co–PI on ODU renewal proposal
  - B. Massoudmoghadam – ODU GRA, MS candidate under MAA

Grant Objectives (Originally proposed in 1991)

- Assess the AQ4 formulation, implementation, and capabilities
- Develop family of elements compatible with AQ4 element (beam and triangular elements)
- Demonstrate the combined use of these elements on a complex structure
- Extend the family of elements to stability, vibration, and geometrically nonlinear problems
- Utilize the GEP in COMET for implementation
- Maintain compatibility with general–purpose FEM code

Grant Objectives (as evolved)

- FIRST YEAR (NAG–1–1374 at Clemson University)
  - Independent assessment of AQ4 4–node shell element
  - Development of compatible 2–node beam
  - Extend quad and beam elements to handle buckling and vibration
- SECOND YEAR (NAG–1–1505 at ODU)
  - Explore alternative stress fields
  - Derive diagonal mass coefficients
  - Further test cases for plates and shells
  - Development of compatible 3–node triangle
  - Explore use of ADR/explicit time integration on MPP systems
- THIRD YEAR (NAG–1–1505 at ODU with Aminpour as Co–PI)
  - Extend element family to handle geometrically nonlinear problems
  - Validate using specific test cases

Outline

- Element Formulation and Approach
- Results for Stress, Buckling and Vibration
- Research Directions
- ADR Performance on MPP Systems
- Future Plans and Summary

Background

- Drilling rotational dof introduced as part of the inplane displacement field – e.g., Allman (1984, 1988), Bergan et al. (1985, 1986), Cook (1986, 1987, 1989), MacNeal and Harder (1988), Yunus et al. (1988, 1989)
- Coupled with the assumed–stress hybrid formulation and Hellinger–Reissner principle in the element development – e.g., Pian (1964, 1984, 1985), Atluri (1984), Cook (1972, 1987), Yunus (1989), Aminpour (1989, 1992)
- Computational framework for finite element methods research and development (COMET, GEP) – Stanley et al. (1990), Knight et al. (1989, 1990), Stewart (1989)

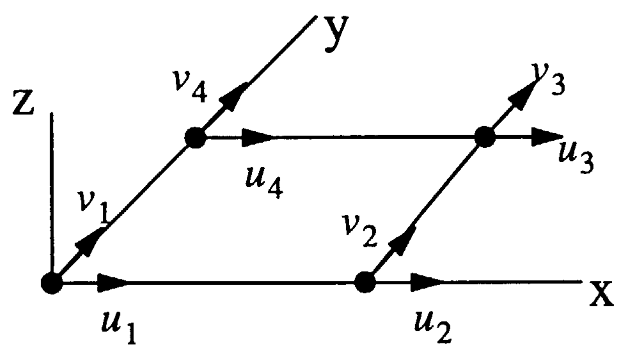
Drilling DOF in Formulation

Two main approaches :

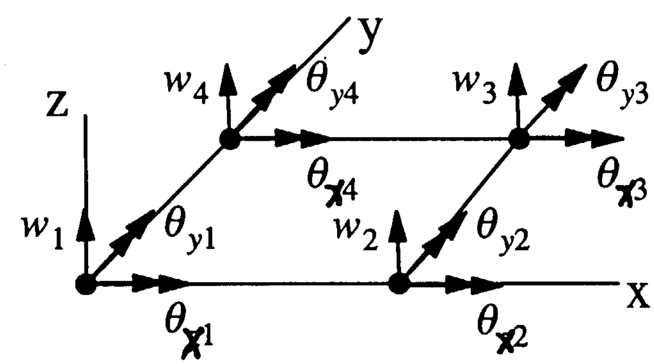
- Drilling Rotations in Displacement Approximations
  - Independent successful attempts by Allman (1984), Bergan and Felippa (1985).
- Independent Rotation Field Included in the Variational Statement
  - First by Reissner (1965), modified by Hughes and Brezzi (1989), Atluri (1984).

Element Degrees–of–Freedom

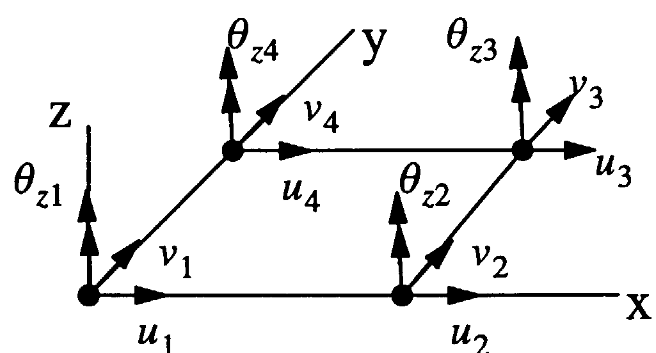
Membrane DoF without Normal Rotations



Bending DoF



Membrane DoF with Normal Rotations

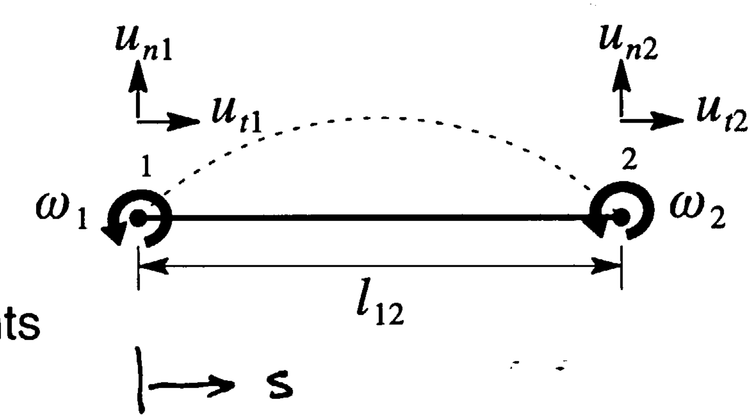


Allman–type Shape Functions

Along an edge

$$u_n = (1 - \frac{s}{l_{12}})\ u_{n1} + (\frac{s}{l_{12}})\ u_{n2} + \frac{1}{2}s\ (1 - \frac{s}{l_{12}})\ (\omega_2 - \omega_1)$$

$$u_t = (1 - \frac{s}{l_{12}})\ u_{t1} + (\frac{s}{l_{12}})\ u_{t2}$$



u_{n1}, u_{n2} : Nodal Normal Displacements
u_{t1}, u_{t2} : Nodal Tangential Displacements
ω_1, ω_2 : Nodal Normal Rotations
l_{12} : Length of the Edge
s : Local Coordinate (varying from 0 to l_{12} along the edge)

Finite Element Approximations

Geometry Approximations (2–node beam):

$$x'(\xi) = \frac{1}{2}(1-\xi)x_1{}' + \frac{1}{2}(1+\xi)x_2{}' = N_1(\xi)x_1{}' + N_2(\xi)x_2{}'$$

Displacement Field Approximations (2–node beam):

$$\begin{Bmatrix} u^{0\prime} \\ v^{0\prime} \\ w^{0\prime} \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} = \begin{Bmatrix} N_1(\xi)u_1^{0\prime} + N_2(\xi)u_2^{0\prime} \\ N_1(\xi)v_1^{0\prime} + \frac{L}{8}(1-\xi^2)\theta_{z1} + N_2(\xi)v_2^{0\prime} + \frac{L}{8}(\xi^2-1)\theta_{z2} \\ N_1(\xi)w_1^{0\prime} + \frac{L}{8}(\xi^2-1)\theta_{y1} + N_2(\xi)w_2^{0\prime} + \frac{L}{8}(1-\xi^2)\theta_{y2} \\ N_1(\xi)\theta_{x1} + N_2(\xi)\theta_{x2} \\ N_1(\xi)\theta_{y1} + N_2(\xi)\theta_{y2} \\ N_1(\xi)\theta_{z1} + N_2(\xi)\theta_{z2} \end{Bmatrix}$$

or

$$\{u^*\} = [\overline{N}(\xi)]_{6\times 12}\{d_e\}_{12\times 1}$$

Finite Element Approximations

Geometry Approximations (4–node quad.):

$$x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)x_i \qquad y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)y_i$$

Displacement Field Approximations (4–node quad.):

$$\{u^*\} = [\overline{N}(\xi,\eta)]_{6\times24}\{d_e\}_{24\times1}$$

Displacement Field Approximations

$$
\begin{aligned}
u^o(\xi,\eta) &\doteq N_i u_i + \frac{\Delta y_i}{8} N_i^*(\theta_{zj} - \theta_{zi}),\\
v^o(\xi,\eta) &= N_i v_i - \frac{\Delta x_i}{8} N_i^*(\theta_{zj} - \theta_{zi}),\\
w^o(\xi,\eta) &= N_i w_i - \frac{\Delta y_i}{8} N_i^*(\theta_{xj} - \theta_{xi}) + \frac{\Delta x_i}{8} N_i^*(\theta_{yj} - \theta_{yi}),\\
\theta_x(\xi,\eta) &= N_i \theta_{xi},\\
\theta_y(\xi,\eta) &= N_i \theta_{yi},
\end{aligned}
$$

Typical Displacement Expansions

$$
N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4,
$$

$$
\begin{aligned}
\frac{\Delta y_i}{8} N_i^*(\theta_{zj} - \theta_{zi}) &= \frac{\Delta y_1}{8} N_1^*(\theta_{z2} - \theta_{z1}) + \frac{\Delta y_2}{8} N_2^*(\theta_{z3} - \theta_{z1})\\
&+ \frac{\Delta y_3}{8} N_3^*(\theta_{z4} - \theta_{z3}) + \frac{\Delta y_4}{8} N_4^*(\theta_{z1} - \theta_{z4}).
\end{aligned}
$$

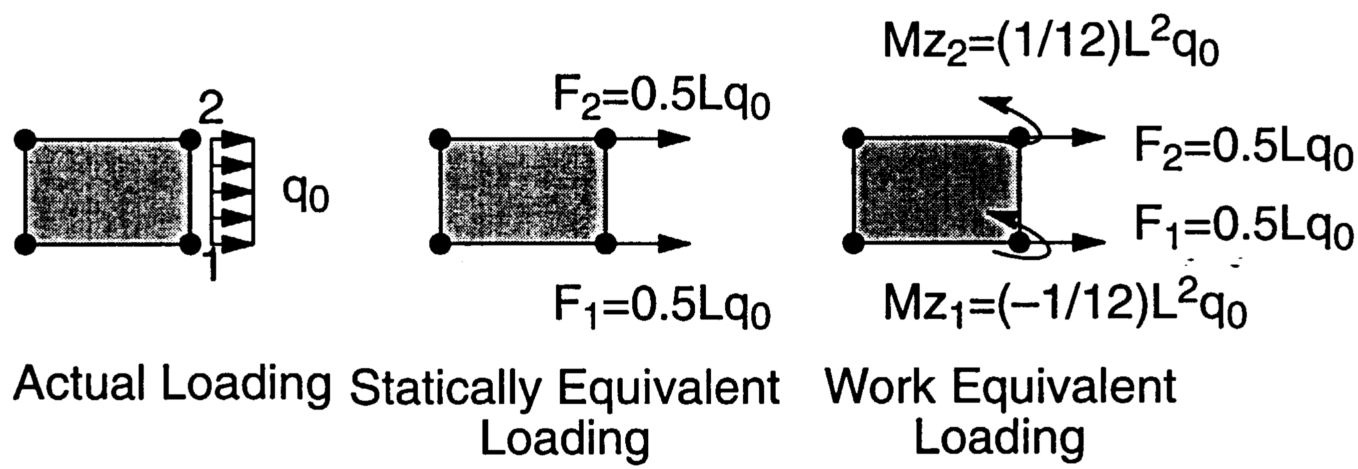
Shape Functions

$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$	$N_1^* = \frac{1}{2}(1-\xi^2)(1-\eta)$
$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$	$N_2^* = \frac{1}{2}(1-\eta^2)(1+\xi)$
$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$	$N_3^* = \frac{1}{2}(1-\xi^2)(1+\eta)$
$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$	$N_4^* = \frac{1}{2}(1-\eta^2)(1-\xi)$

Effect on Element Loads

- Derivation of work equivalent, consistent loads includes normal rotations

- Affects stress distribution locally with minor effect on displacements



NAG1-1505

Assumed–Stress Hybrid Elements

- First introduced by Pian (1964), later pioneered by Pian and his co–workers.
- Initially based on a modified form of complementary energy principle, but now mostly based on Hellinger–Reissner variational principle.
- Displacements described throughout the element. (including the boundaries)
- Stresses described only in interior of the element. (no interelement stress continuity)

Hellinger–Reissner Functional

In Vector Form :

$$\pi_{HR} = -\frac{1}{2}\int_A \{\sigma *\}^T[D *]\{\sigma *\}\ dA + \int_A \{\sigma *\}^T[\mathcal{L} *]\{u *\}\ dA - \int_{S_\sigma} \{u *\}^T[R^*]^T\{t_o\}\ dS$$

Where,

$$\{\sigma *\} = \left\{N_x\ N_y\ N_{xy}\ M_x\ M_y\ M_{xy}\ Q_x\ Q_y\right\}^T$$

$$\{u *\} = \left\{u^o\ v^o\ w^o\ \theta_x\ \theta_y\ \theta_z\right\}^T$$

Field Approximations

- Stress Approximation

$$\{\sigma *\} = [P]\ \{\beta\}$$

- Displacement Approximation

$$\{u *\} = [N]\ \{q\}$$

Upon substitution, the functional reduces to

$$\pi_{HR} = -\frac{1}{2}\{\beta\}^T[H]\{\beta\} + \{\beta\}^T[T]\{q\} - \{q\}^T\{F\}$$

Where,

$$[H] = \int_A [P]^T[D *][P]\ dA \qquad [T] = \int_A [P]^T[\mathcal{L} *][N]\ dA$$

$$\{F\} = \int_{S_\sigma} [N]^T[R *]\{t_o\}\ dS$$

Element Stiffness Matrix

Imposing Stationary Conditions on the Functional

$$\delta\pi_{HR} = \frac{\partial\pi_{HR}}{\partial\{\beta\}}\,\delta\{\beta\} + \frac{\partial\pi_{HR}}{\partial\{q\}}\,\delta\{q\} = 0$$

$$\frac{\partial\pi_{HR}}{\partial\{\beta\}} = 0 \quad \Longrightarrow \quad \{\beta\} = [H]^{-1}[T]\{q\}$$

Substituting this back in the functional, and then

$$\frac{\partial\pi_{HR}}{\partial\{q\}} = 0 \quad \Longrightarrow \quad [k_e]\{q\} = \{F\}$$

where,

$$[k_e] = [T]^T[H]^{-1}[T]$$

is the Element Stiffness Matrix

Symbolic Computations

Derivation of the beam elemental matrices and arrays was performed using symbolic computational methods (i.e., MAPLE). That is, the operations needed for

$$[H] = \int_0^L [P]^T [D^*] [P] dx = \int_{-1}^{+1} [P]^T [D^*] [P] \frac{L}{2} d\xi$$

are performed symbolically using the following MAPLE commands:

```
DstarP:=multiply(Dstar,P):
PTDstarP:=multiply(PT,DstarP):
H:=array(1..6,1..6):
for i from 1 to 6 do
for j from 1 to 6 do
H[i,j]:=simplify((L/2)*int(PTDstarP[i,j],XI=-1..1)):
fortran(H[i,j],optimized);
od:
od:
```

Symbolic Computations, cont.

Derivation of the shell elemental matrices and arrays was also performed using symbolic methods but are more complicated.

$$[H] = \int_A [P]^T [D^*] \; [P] \; dA$$

```
Estar:=array(1..8,1..8):
P:=array([ [1,0,0,xi,0,eta,0,eta*eta,0,0,0,0,0,0,0,0,0,0,0,0,0,0],
[0,1,0,0,xi,0,eta,0,xi*xi,0,0,0,0,0,0,0,0,0,0,0,0,0],
[0,0,1,-eta,0,0,-xi,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],
[0,0,0,0,0,0,0,0,0,1,0,0,xi,0,eta,0,eta*eta,0,0,0,0,0],
[0,0,0,0,0,0,0,0,0,0,1,0,0,xi,0,eta,0,xi*xi,0,0,0,0],
[0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,xi,eta,0.5*(xi*xi),0.5*(eta*eta)],
[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,eta],
[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,xi,0] ]);
PT:=transpose(P):
EP:=multiply(Estar,P):
PTEP:=multiply(PT,EP):
H:=array(symmetric,1..22,1..22):
JAC:=a1*xi+a2*eta+a3;
for i from 1 to 22 do
for j from i to 22 do
H[i,j]:=simplify(int(int(PTEP[i,j]*JAC,eta=-1..1),xi=-1..1)):
od:
od:
```

Symbolic Computations, cont.

Typical MAPLE–generated coefficients of *[H]* and *[T]* matrices:

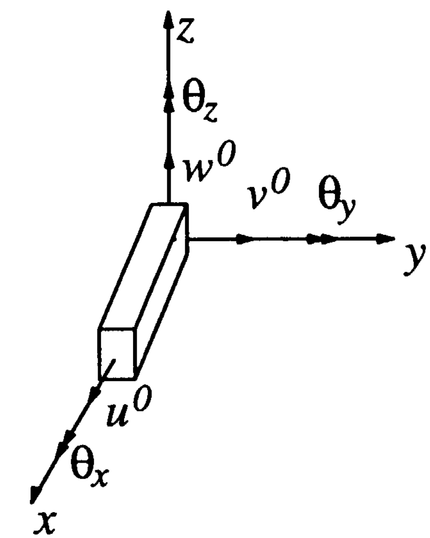
```
 H(22,22) = 0.2D0*Estar(6,6)*a3+0.1333333333D1*Estar(7,7)*a3+0.4D0*
#Estar(6,7)*a2+0.4D0*Estar(7,6)*a2
```

```
 T(4,18) = 4.D0/9.D0*aJACt(1,3)*P178+4.D0/3.D0*aJACt(1,2)*p232+4.D0
#/3.D0*aJACt(1,3)*p152+4.D0/9.D0*aJACt(1,1)*P118+4.D0/9.D0*aJACt(1,
#3)*P158+4.D0/9.D0*aJACt(1,2)*P238+4.D0/3.D0*aJACt(1,1)*p112+4.D0/3
#.D0*aJACt(1,3)*p192-4.D0/9.D0*aJACt(3,1)*p117-4.D0/9.D0*aJACt(3,3)
#*p197-4.D0/9.D0*aJACt(3,2)*p237-4.D0/9.D0*aJACt(3,3)*p157-4.D0/3.D
#0*aJACt(3,2)*p233-4.D0/3.D0*aJACt(3,3)*p193-4.D0/3.D0*aJACt(3,3)*p
#153-4.D0/3.D0*aJACt(3,1)*p113
```

3–D Beam Theory

Kinematics:

$$
\begin{aligned}
u(x,y,z) &= u^0(x) + z\,\theta_y(x) - y\,\theta_z(x) \\
v(x,y,z) &= v^0(x) - z\,\theta_x(x) \\
w(x,y,z) &= w^0(x) + y\,\theta_x(x)
\end{aligned}
$$



or

$$
\{u\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & 0 \\ 0 & 0 & 1 & y & 0 & 0 \end{bmatrix} \begin{Bmatrix} u^0 \\ v^0 \\ w^0 \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} = \left[R^*\right]\left\{u^*\right\}
$$

3–D Beam Theory

Strain–Displacement Relations:

$$\varepsilon_x(x,y,z) = \frac{\partial u}{\partial x} = \frac{\partial u^0}{\partial x} + z\frac{\partial \theta_y}{\partial x} + y\,(-\frac{\partial \theta_z}{\partial x})$$

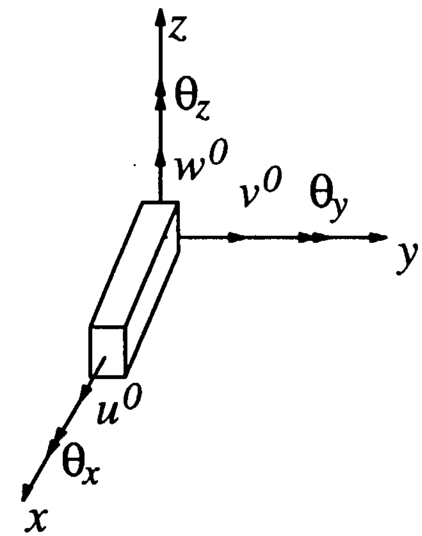
$$= \varepsilon_x^0(x) + z\,\varkappa_y(x) + y\,\varkappa_z(x)$$

$$\gamma_{xy}(x,y,z) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = (-\theta_z + \frac{\partial v^0}{\partial x}) - z\frac{\partial \theta_x}{\partial x}$$

$$= \gamma_{xy}^0(x) - z\,\alpha(x)$$

$$\gamma_{xz}(x,y,z) = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = (\theta_y + \frac{\partial w^0}{\partial x}) + y\frac{\partial \theta_x}{\partial x}$$

$$= \gamma_{xz}^0(x) + y\,\alpha(x)$$



3–D Beam Theory

Strain–Displacement Relations, continued:

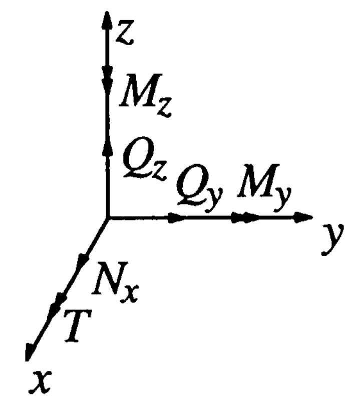
$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} 1 & z & y & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & 1 & 0 \\ 0 & 0 & 0 & y & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varkappa_y \\ \varkappa_z \\ \alpha \\ \gamma_{xy}^0 \\ \gamma_{xz}^0 \end{Bmatrix} = [R]\{\varepsilon^*\}$$

and

$$\{\varepsilon^*\} = \begin{bmatrix} \partial_x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_x & 0 \\ 0 & 0 & 0 & 0 & 0 & -\partial_x \\ 0 & 0 & 0 & \partial_x & 0 & 0 \\ 0 & \partial_x & 0 & 0 & 0 & -1 \\ 0 & 0 & \partial_x & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u^0 \\ v^0 \\ w^0 \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} = [\mathcal{L}^*]\{u^*\}$$

3–D Beam Theory

Stress–Strain Relations:



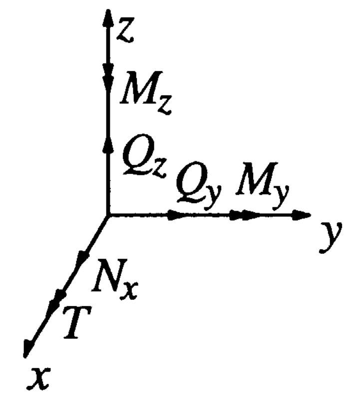
$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \tau_{xy} \\ \tau xz \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} = [C]\{\varepsilon\}$$

Determine the beam force and moment resultants:

$$\{\sigma^*\} = \int_A [R]^T \{\sigma\} dA = \left[\int_A [R]^T [C][R] dA\right] \{\varepsilon^*\} = [C^*]\{\varepsilon^*\}$$

3–D Beam Theory

Stress–Strain Relations, continued:



$$\{\sigma^*\} = [C^*]\{\varepsilon^*\}$$

where

$$\begin{Bmatrix} N_x \\ M_y \\ M_z \\ T \\ Q_y \\ Q_z \end{Bmatrix} = \begin{bmatrix} EA & EAe_z & EAe_y & 0 & 0 & 0 \\ EAe_z & EI_{zz} & EI_{yz} & 0 & 0 & 0 \\ EAe_y & EI_{yz} & EI_{yy} & 0 & 0 & 0 \\ 0 & 0 & 0 & GJ & -GA_s e_z & GA_s e_y \\ 0 & 0 & 0 & -GA_s e_z & GA_s & 0 \\ 0 & 0 & 0 & GA_s e_y & 0 & GA_s \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varkappa_y \\ \varkappa_z \\ \alpha \\ \gamma_{xy}^0 \\ \gamma_{xz}^0 \end{Bmatrix}$$

and $\{\varepsilon^*\} = [C^*]^{-1}\{\sigma^*\} = [D^*]\{\sigma^*\}$ from LAUB

Finite Element Approximations

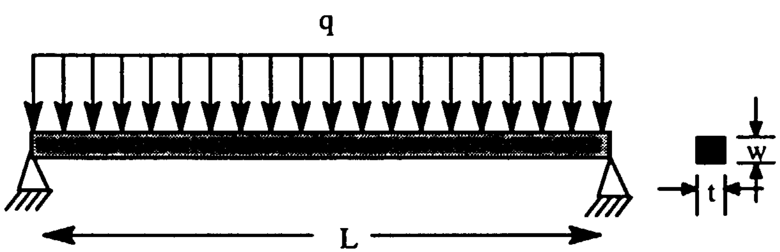
Stress Field Approximations (2–node beam):

12 displacement dof/element, 6 rigid body modes

Stress field needs a minimum of 6 parameters

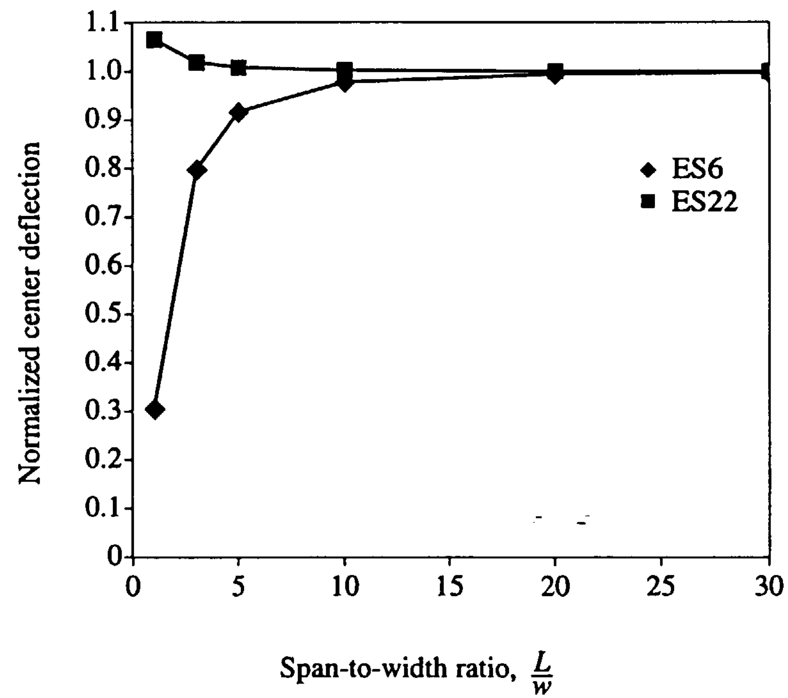
$$\begin{Bmatrix} N_x \\ M_y \\ M_z \\ T \\ Q_y \\ Q_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \xi \\ 0 & 0 & 1 & 0 & \xi & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/L & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/L \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{Bmatrix} \quad \text{or} \quad \{\sigma^*\} = [P]_{6\times 6}\{\beta\}_{6\times 1}$$

Beam Bending

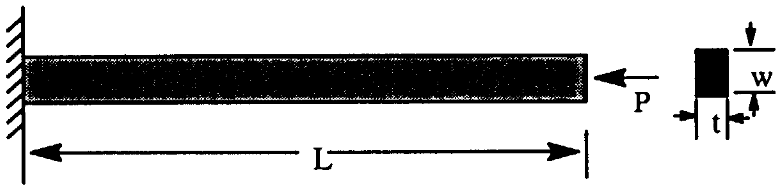


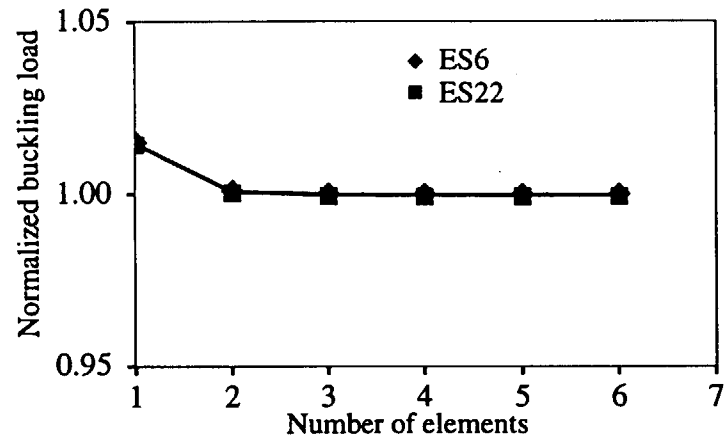
<u>Properties of the beam.</u>

E	=	1 . x 10^7psi,
ν	=	0 . 3,
L	=	10 . in,
t	=	1 . 0 in,
q	=	1 . 0 lb/in .



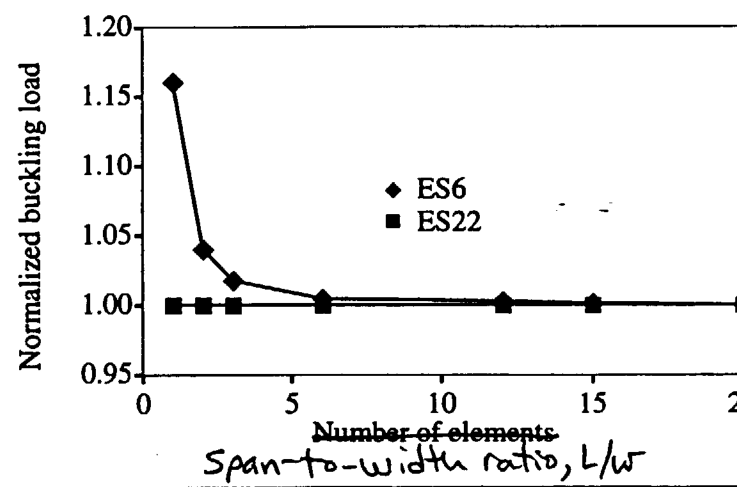
Beam Buckling





Properties of the Beam (consistent units)

E	= 1 . x 10^7
ν	= 0 . 3
L	= 6 . 0
w	= 0 . 2
t	= 0 . 1
P	= 1 . 0



Deformation Modes

- Membrane

  12 Deformation Modes = 3 Rigid Body Modes
  + 3 Constant Strain States
  + 5 Higher–Order Strain States
  + 1 Spurious Zero–Energy Mode

- Bending

  12 Deformation Modes = 3 Rigid Body Modes
  + 5 Constant Strain States
  + 4 Higher–Order Strain States

- Note : θ_z not interpolated independently

Finite Element Approximations

Stress Field Approximations (4–node quad.):

24 displacement dof/element, 6 rigid body modes

Stress field needs a *minimum* of 18 parameters

(AQ4 has 9 for membrane; 13 for bending)

$$\begin{Bmatrix} N_\xi \\ N_\eta \\ N_{\xi\eta} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \eta & 0 & \xi & 0 & \eta^2 & 0 \\ 0 & 1 & 0 & 0 & \xi & 0 & \eta & 0 & \xi^2 \\ 0 & 0 & 1 & 0 & 0 & -\eta & -\xi & 0 & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \end{Bmatrix} \qquad \begin{Bmatrix} M_\xi \\ M_\eta \\ M_{\xi\eta} \\ Q_\xi \\ Q_\eta \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \eta & 0 & \xi & 0 & \eta^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \xi & 0 & \eta & 0 & \xi^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & \eta & \frac{1}{2}\xi^2 & \frac{1}{2}\eta^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \eta \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \xi & 0 \end{bmatrix} \begin{Bmatrix} \bar{\beta}_1 \\ \bar{\beta}_2 \\ \bar{\beta}_3 \\ \bar{\beta}_4 \\ \bar{\beta}_5 \\ \bar{\beta}_6 \\ \bar{\beta}_7 \\ \bar{\beta}_8 \\ \bar{\beta}_9 \\ \bar{\beta}_{10} \\ \bar{\beta}_{11} \\ \bar{\beta}_{12} \\ \bar{\beta}_{13} \end{Bmatrix}$$

$$\left\{\sigma^*\right\} = [P]_{8\times 22}\{\beta\}_{22\times 1}$$

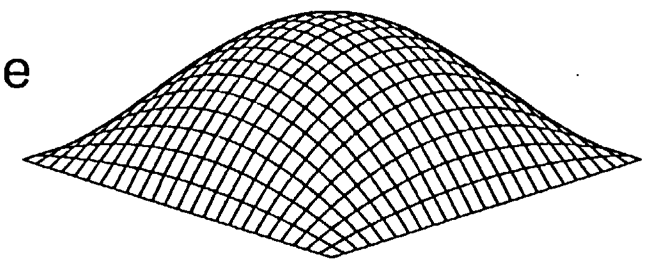
Improved Displacement Field

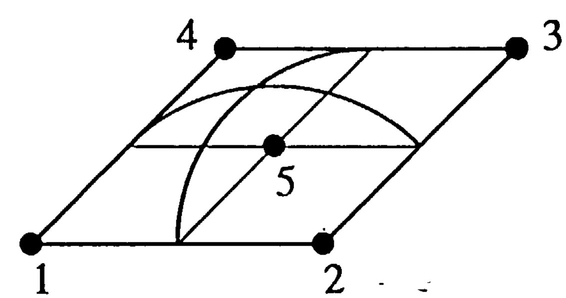
- Introduction of a Bubble Function
- Bubble Function corresponds to a Node at the Center of the Element

$$N_5 = (1 - \xi^2)\ (1 - \eta^2)$$

- Additional Degrees of Freedom

$$u_5, v_5, w_5, \theta_{x5}, \theta_{y5}\ (no\ \theta_{z5})$$





- Stiffness Matrix condensed to retain the original order of the Element Stiffness Matrix (24 x 24)

Family of Hybrid Shell Elements

Element name	Additional Modifications	Number of displacement *dof*		Number of stress parameters	
		In-Plane	Out-of-Plane	Membrane	Bending
A4S1	Symbolic Version of A4N1/AQ4	12	12	9	13
A4S2	Modified Membrane Stress Field	12	12	11	13
A4S3	Bubble Functions (Out-of-plane)	12	15	9	17
A4S4	Bubble Functions (In-plane)	14	12	11	13
A4S5	Bubble Functions (Both)	14	15	11	17
A4S6	Bubble Functions (Out-of-plane)	12	15	9	19

Membrane Stress Field

$$\boxed{\begin{array}{lcl} N_\xi & = & \beta_1 + \beta_4\xi + \beta_6\eta + \beta_8\eta^2 \\ N_\eta & = & \beta_2 + \beta_5\xi + \beta_7\eta + \beta_9\xi^2 \\ N_{\xi\eta} & = & \beta_3 - \beta_4\eta - \beta_7\xi \end{array}}$$

Bending Stress Field

$$\boxed{\begin{array}{lcl} M_\xi & = & \bar{\beta}_1 + \bar{\beta}_4\xi + \bar{\beta}_6\eta + \bar{\beta}_8\eta^2 \\ M_\eta & = & \bar{\beta}_2 + \bar{\beta}_5\xi + \bar{\beta}_7\eta + \bar{\beta}_9\xi^2 \\ M_{\xi\eta} & = & \bar{\beta}_3 + \bar{\beta}_{10}\xi + \bar{\beta}_{11}\eta + \frac{1}{2}\bar{\beta}_{12}\xi^2 + \frac{1}{2}\bar{\beta}_{13}\eta^2 \end{array}}$$

Transverse Shear Stress Field

$$\boxed{\begin{array}{lcl} Q_\xi & = & \bar{\beta}_4 + \bar{\beta}_{11} + \bar{\beta}_{13}\eta \\ Q_\eta & = & \bar{\beta}_7 + \bar{\beta}_{10} + \bar{\beta}_{12}\xi \end{array}}$$

Modified Stress Field

Proposed Membrane Stress Field

$$\boxed{\begin{aligned} N_\xi &= \beta_1 + \beta_4\xi + \beta_6\eta + \beta_8\eta^2 + \beta_{10}\xi\eta \\ N_\eta &= \beta_2 + \beta_5\xi + \beta_7\eta + \beta_9\xi^2 + \beta_{11}\xi\eta \\ N_{\xi\eta} &= \beta_3 - \beta_4\eta - \beta_7\xi - \tfrac{1}{2}\beta_{10}\eta^2 - \tfrac{1}{2}\beta_{11}\xi^2 \end{aligned}}$$

<u>Remarks</u> :

- 11 independent stress parameters to suppress 11 independent deformation modes (9 + 2 due to bubble function)
- Equilibrium equations satisfied exactly on specializing field to Cartesian basis
- Field produces rank deficiency
- Equivalent to enforcing equilibrium in variational statement (using Lagrange multipliers)

Remedy

$\boxed{\textit{Not satisfying equilibrium equations a priori}}$

How?

1. Additional terms in the approximations
2. Uncoupled stress field approximations

Alternate Stress Fields

Membrane

$$\boxed{\begin{array}{lcl} N_\xi & = & \beta_1 + \beta_4\xi + \beta_7\eta + \beta_{10}\eta^2 \\ N_\eta & = & \beta_2 + \beta_5\xi + \beta_8\eta + \beta_{11}\xi^2 \\ N_{\xi\eta} & = & \beta_3 + \beta_6\xi + \beta_9\eta \end{array}}$$

Transverse Shear I (*additional terms*)

$$\boxed{\begin{array}{lcl} Q_\xi & = & \bar{\beta}_4 + \bar{\beta}_{11} + \bar{\beta}_{13}\eta + \bar{\beta}_{14} + \bar{\beta}_{15}\xi \\ Q_\eta & = & \bar{\beta}_7 + \bar{\beta}_{10} + \bar{\beta}_{12}\xi + \bar{\beta}_{16}\eta + \bar{\beta}_{17} \end{array}}$$

Transverse Shear II (*uncoupled field*)

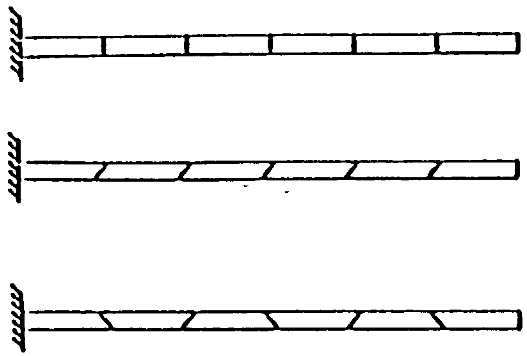
$$\boxed{\begin{array}{lcl} Q_\xi & = & \bar{\beta}_{14} + \bar{\beta}_{16}\xi + \bar{\beta}_{18}\eta \\ Q_\eta & = & \bar{\beta}_{15} + \bar{\beta}_{17}\xi + \bar{\beta}_{19}\eta \end{array}}$$

Note: Bending stress field remains the same

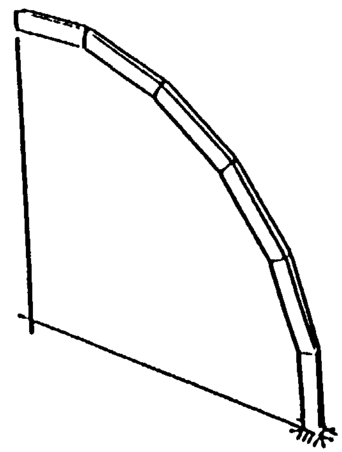
AQ4 Shell Element Assessment

- Replicated the element test cases reported by Aminpour
- Performed additional bending test cases for mesh distortion and shear locking
- Performed additional shell analysis using the pear–shaped cylinder test case

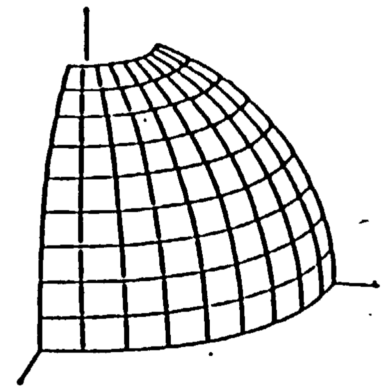
MACNEAL-HARDER PROBLEMS



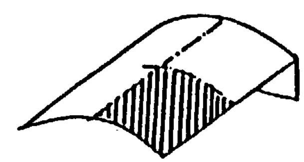
CANTILEVER BEAM



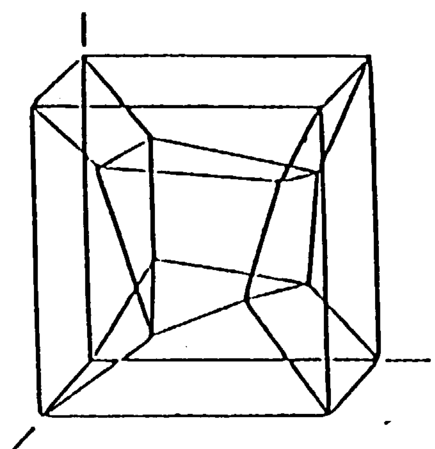
CURVED BEAM



SPHERICAL SHELL

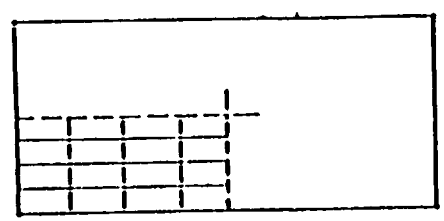


SCORDELIS-LO
ROOF



PATCH TEST

TWISTED BEAM



RECTANGULAR
PLATE

THICK-WALLED
CYLINDER

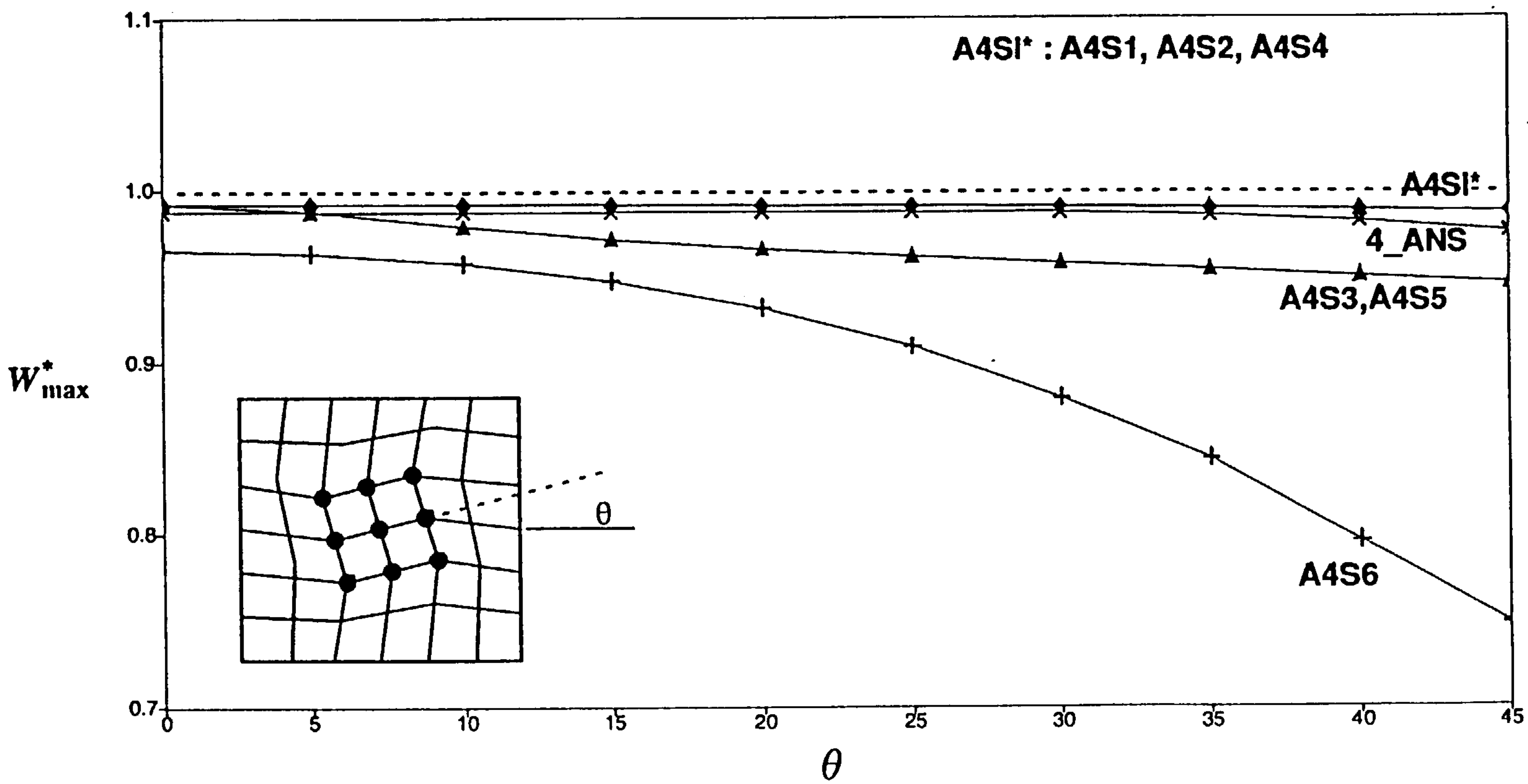
M–H Cantilever Beams

A : Extension, B : In-plane Shear, C : Out-of-plane Shear, D : Twist									
Load	4_ANS	4_MSC	4_STG	A4S1	A4S2	A4S3	A4S4	A4S5	A4S6
Rectangular-Shaped Elements									
A	0.995	0.995	0.994	0.998	0.998	0.998	0.988	0.988	0.988
B	0.904	0.904*	0.915	0.993	0.993	0.993	0.993	0.993	0.993
C	0.980	0.986	0.986	0.981	0.981	0.981	0.981	0.981	0.981
D	0.856	0.941	0.680	1.009	1.009	1.009	1.009	1.009	0.858
Trapezoidal-Shaped Elements									
A	0.761	0.996	0.991	0.998	0.998	0.998	0.998	0.998	0.998
B	0.305	0.071*	0.813	0.986	0.985	0.986	0.986	0.986	0.986
C	0.763	0.968	†	0.969	0.969	0.968	0.969	0.968	0.961
D	0.843	0.951	†	1.007	1.007	1.004	1.007	1.004	0.856
Parallelogram-Shaped Elements									
A	0.966	0.996	9.989	0.998	0.998	0.998	0.998	0.998	0.998
B	0.324	0.080*	0.794	0.977	0.972	0.977	0.977	0.977	0.977
C	0.939	0.977	0.991	0.980	0.980	0.980	0.980	0.980	0.979
D	0.798	0.945	0.677	1.007	1.007	0.999	1.007	0.999	0.846

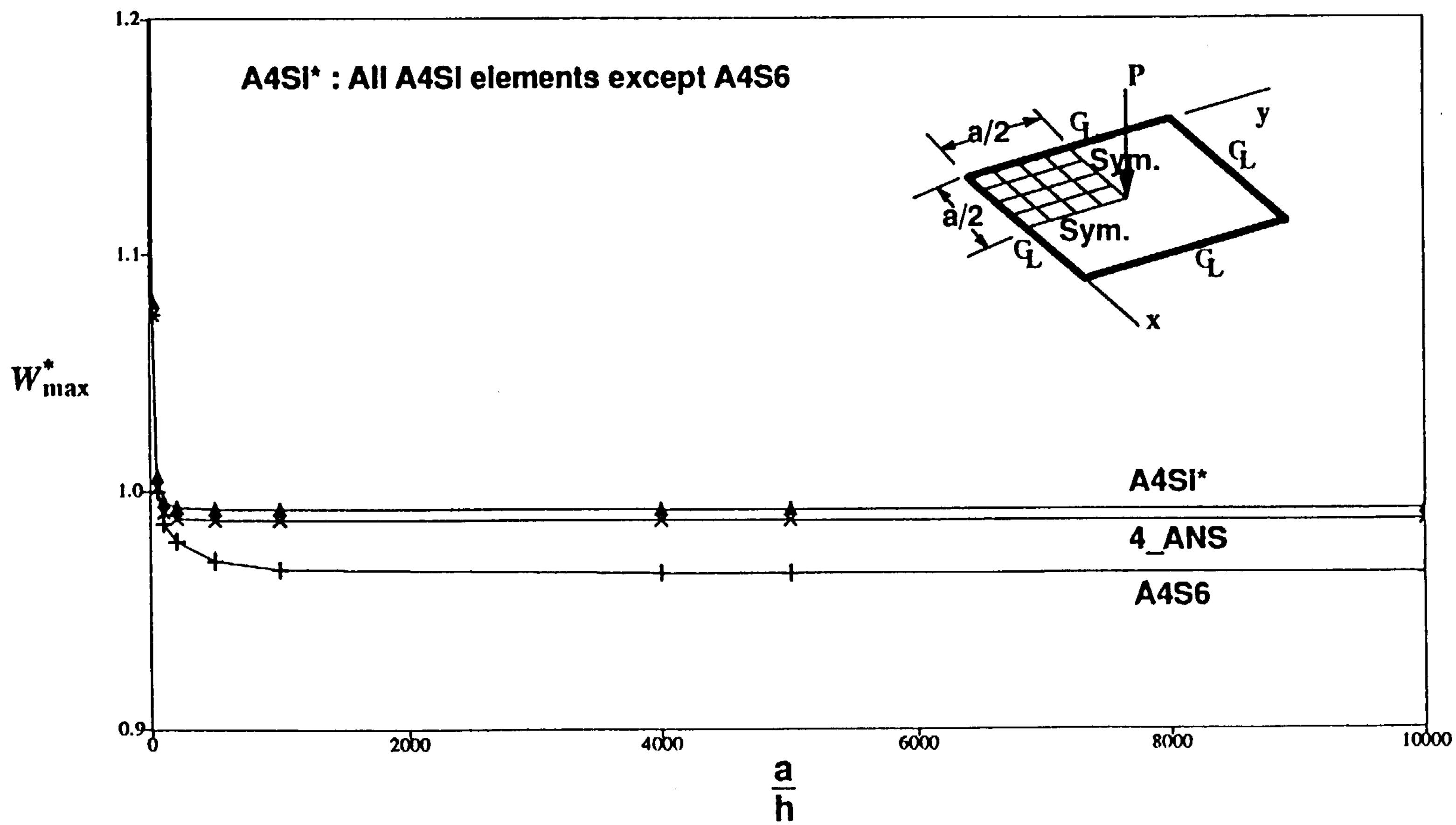
* Q4S results for these cases are 0.993, 0.988, and 0.986, respectively.

† Produces a singular stiffness matrix.

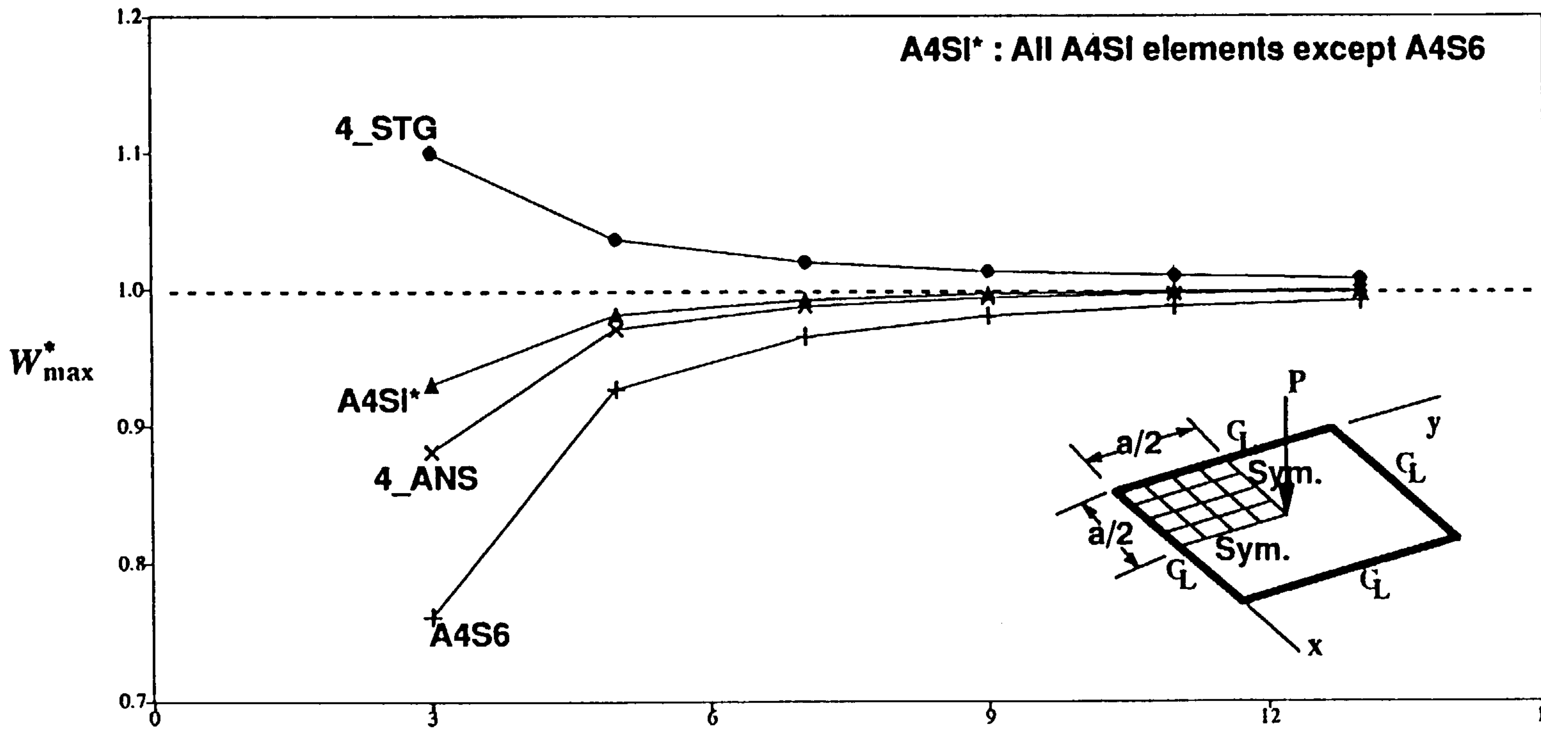
Effect of Mesh Distortion



Effect of Changing Thickness



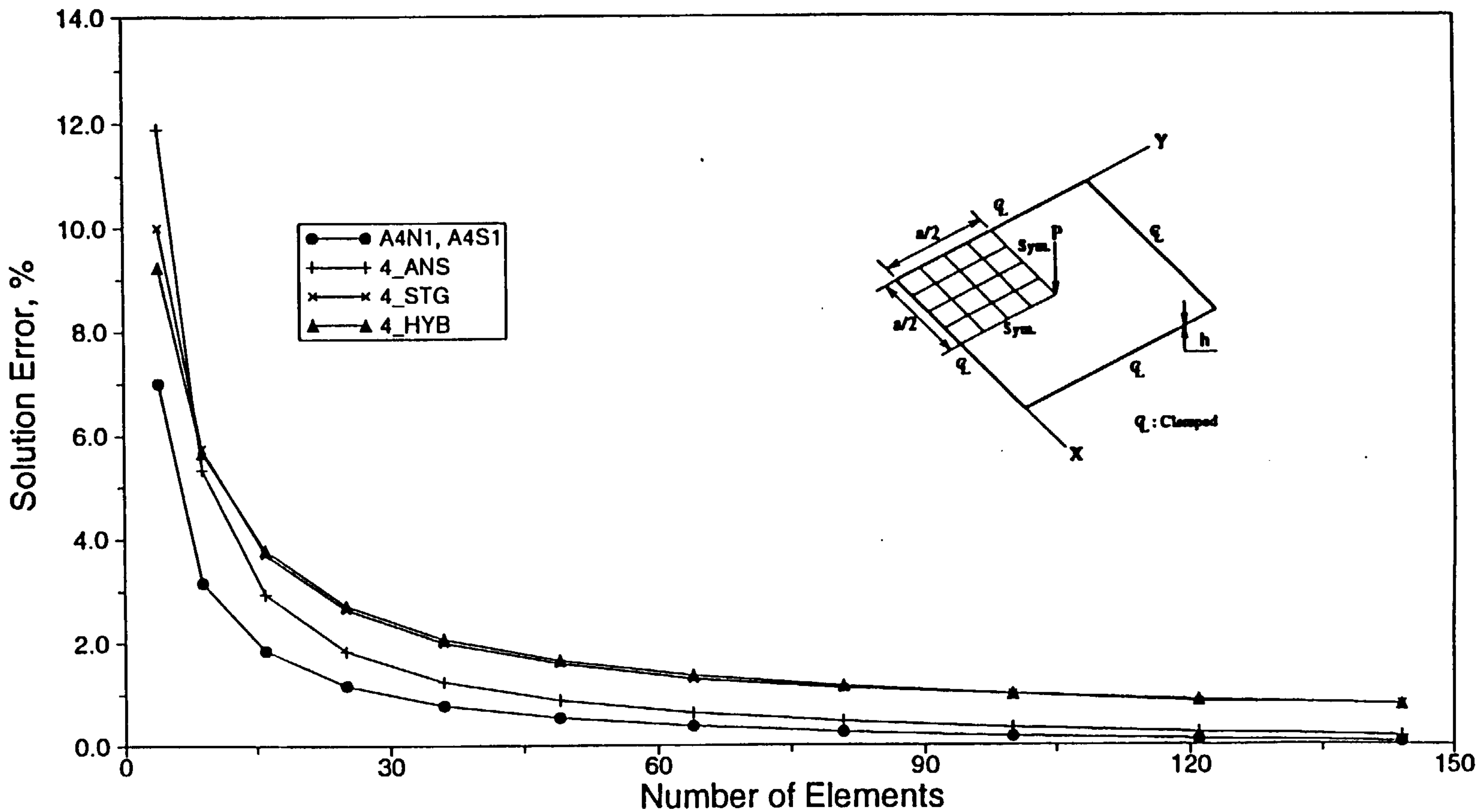
Convergence



N x N

(N : Number of nodes per side)

Solution Error per Element



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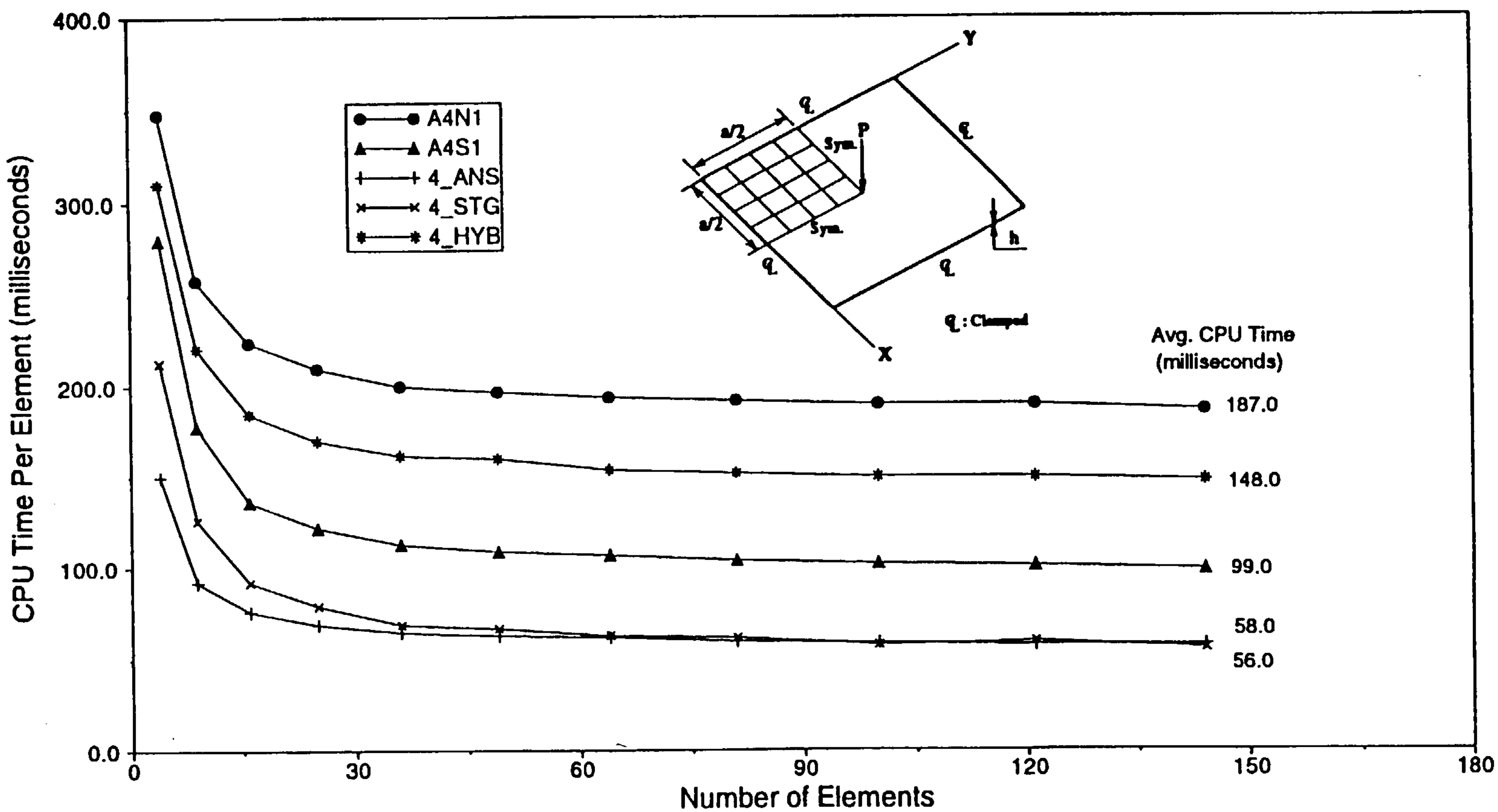
CPU Time Per Element



Timing for Element Family

Isotropic Square Plate

144 Elements (13×13 mesh)	
Element	Normalized CPU Time per Element
A4S1	1.000
A4S2	1.054
A4S3	1.789
A4S4	1.640
A4S5	1.917
A4S6	1.799
A4N1	1.887

Computational Effort Required for Specified Solution Error

Solution Error	Element Name	Number of elements	t^K/t^K_{A4S1}	t^O/t^O_{A4S1}
< 5 %	A4S1	9	1.000	1.000
	A4N1	9	1.450	1.044
	4_ANS	16	0.763	0.835
	4_STG	16	0.919	1.098
	4_HYB	16	1.850	1.146
< 2 %	A4S1	16	1.000	1.000
	A4N1	16	1.650	1.114
	4_ANS	25	0.793	0.883
	4_STG	36	1.147	1.375
	4_HYB	49	3.622	1.738
< 1 %	A4S1	36	1.000	1.000
	A4N1	36	1.783	1.198
	4_ANS	49	0.758	0.891
	4_STG	100	1.440	1.978
	4_HYB	121	4.486	2.499
< 0.5 %	A4S1	64	1.000	1.000
	A4N1	64	1.809	1.269
	4_ANS	81	0.706	0.884
	4_STG	–	–	–
	4_HYB	–	–	–

t^K : Total CPU time to evaluate all the element stiffness matrices.

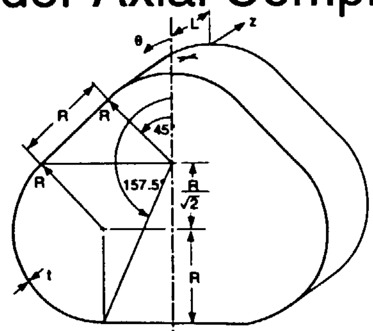
t^O : Overall solution time (total time from start to finish).

t^K_{A4S1} : Total CPU time to evaluate all the element stiffness matrices using A4S1.

t^O_{A4S1} : Overall solution time using A4S1.

Application of Shell Element

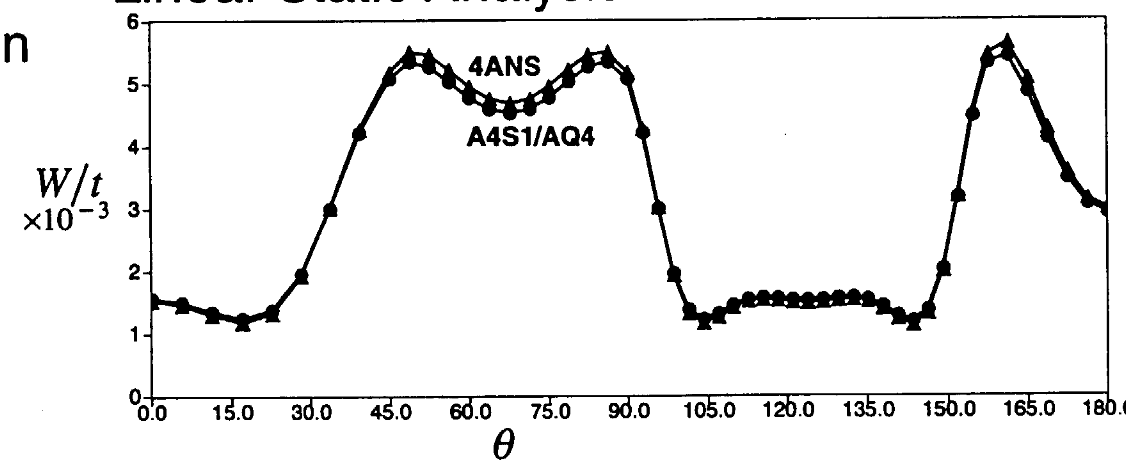
Pear–shaped Cylinder Under Axial Compression



Material Properties:
$E = 10^7$ psi (68.95 GPa)
$\nu = 0.3$

Geometric Parameters:
R = 1.0 inch (25.40 mm)
t = 0.01 inches (0.254 mm)
L = 0.8 inches (20.32 mm)

Linear Static Analysis – Normal Deflections

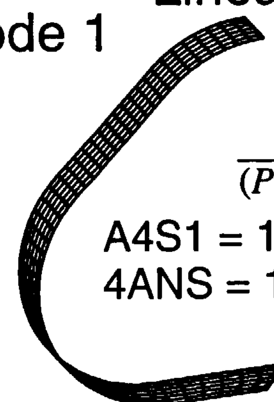


Element Stiffness Matrix Formation Time

Element	Relative CPU Times
AQ4	1.00
A4S1	0.52
4ANS	0.30
E410	0.30

Linear Buckling Analysis

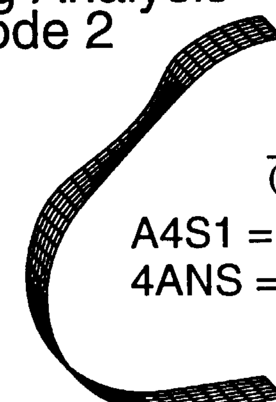
Mode 1



$$\frac{P}{(P_{cr})_{converged}}$$

A4S1 = 1.013
4ANS = 1.021

Mode 2

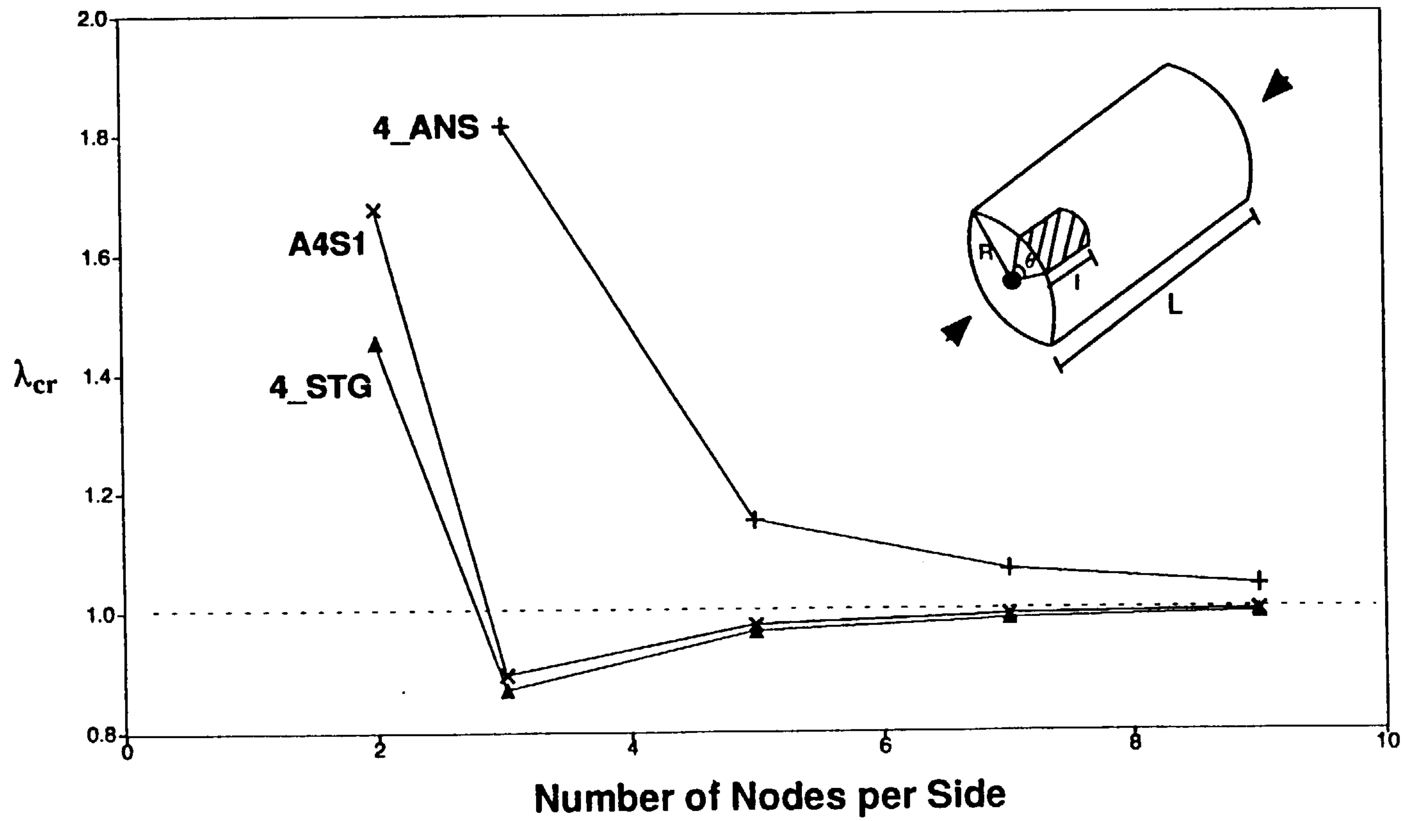


$$\frac{P}{(P_{cr})_{converged}}$$

A4S1 = 1.016
4ANS = 1.032

Axially Compressed Cylinder

Convergence of Buckling Load



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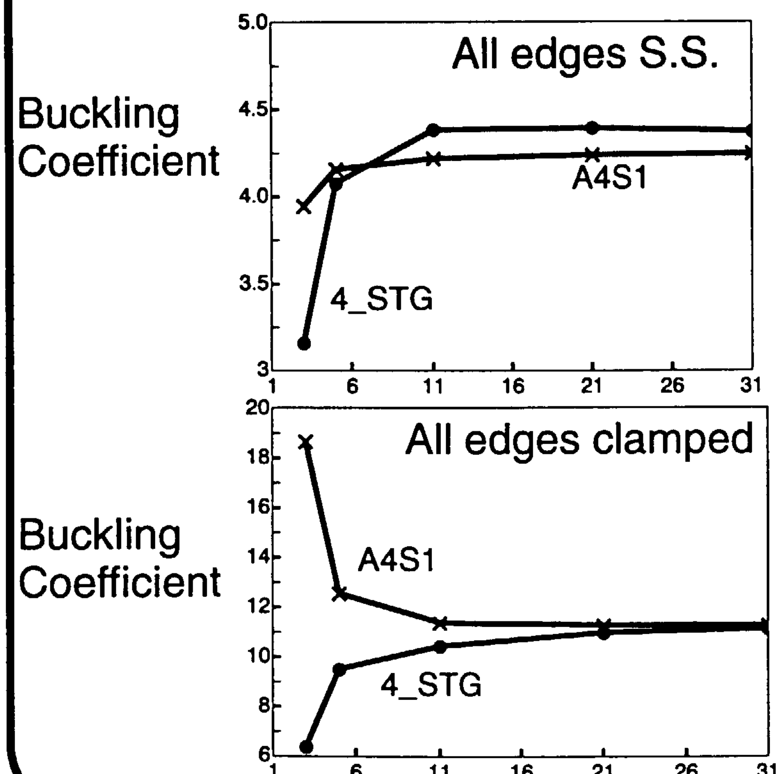
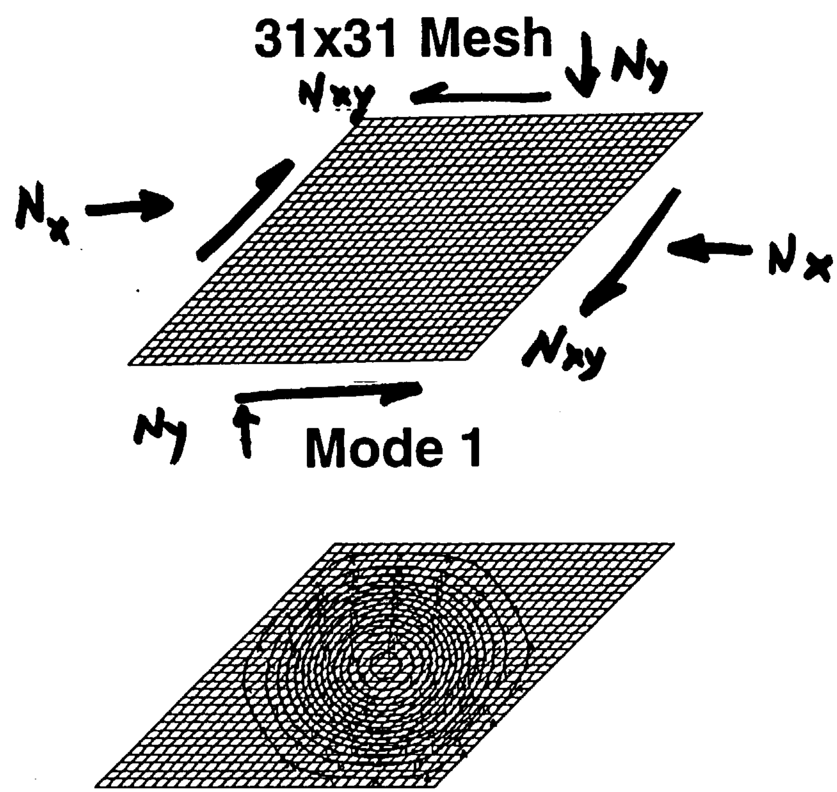
Skewed Laminated Plate

Convergence of Buckling Load

$[\pm 45/90/0]_S$ Laminate; Combined in–plane loading





Explicit Time Integration and ADR

- A technique for solving the semi-discrete equations of motion

$$\mathbf{M}\ddot{\mathbf{D}} + \mathbf{C}\dot{\mathbf{D}} + \mathbf{F}(\mathbf{D}) = \mathbf{P}$$

- Use explicit time integration scheme such as Central Difference

$$\dot{\mathbf{D}}^n = \frac{1}{2h}(\mathbf{D}^{n+1} - \mathbf{D}^{n-1}) \qquad \ddot{\mathbf{D}}^n = \frac{1}{h^2}(\mathbf{D}^{n+1} - 2\mathbf{D}^n + \mathbf{D}^{n-1})$$

- Use diagonal $\mathbf{M}$ and mass-proportional damping $\mathbf{C} = c\mathbf{M}$

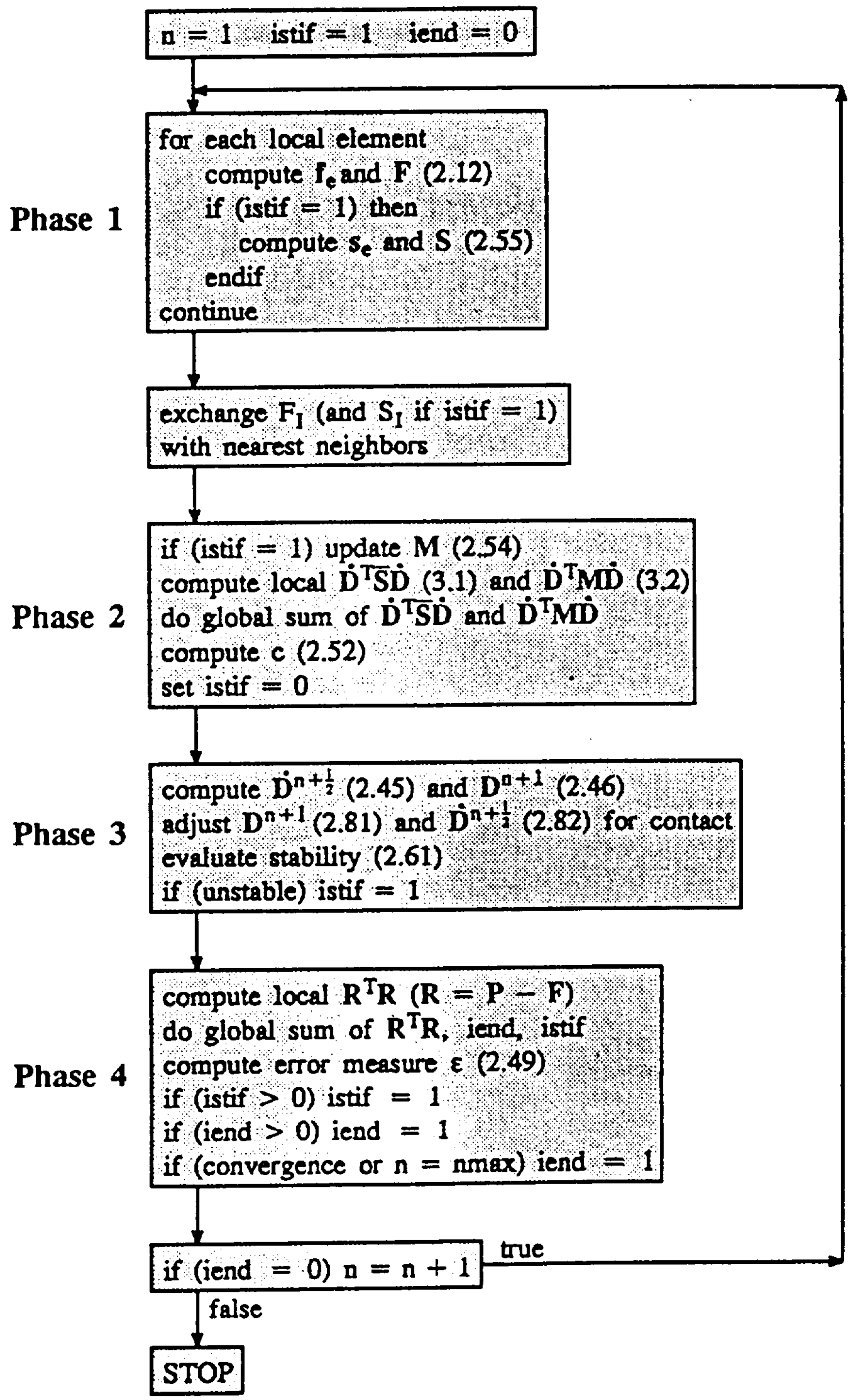
- Resulting fundamental time-marching equation becomes

$$\mathbf{D}^{n+1} = \left(\frac{2h^2}{2+ch}\right)\mathbf{M}^{-1}(\mathbf{P}^n - \mathbf{F}^n) + \left(\frac{4}{2+ch}\right)\mathbf{D}^n - \left(\frac{2-ch}{2+ch}\right)\mathbf{D}^{n-1}$$

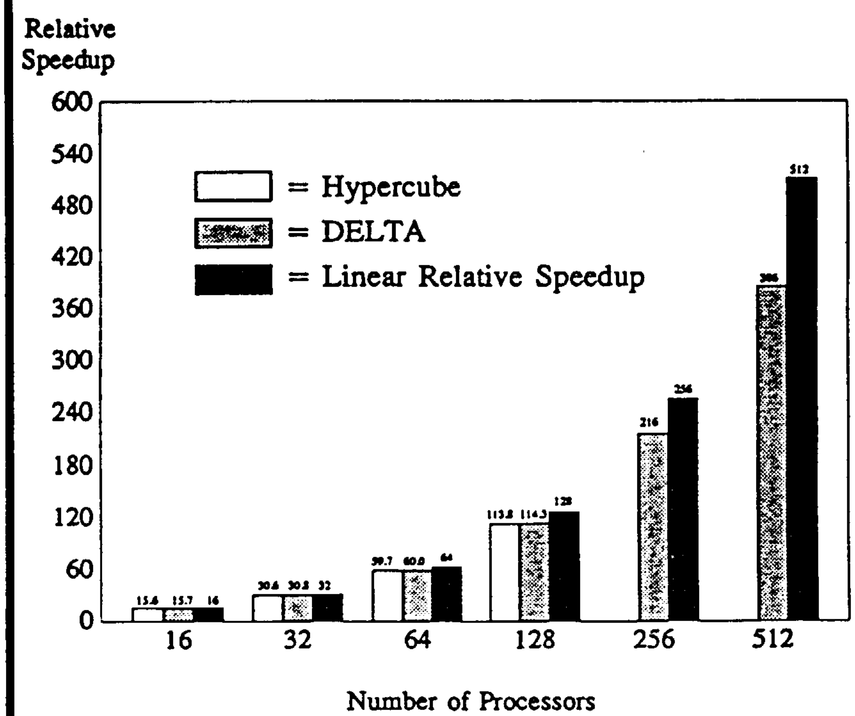
- For a given time step, most of computational effort is in evaluation of $\mathbf{F}^n$ (all other quantities on RHS are known)

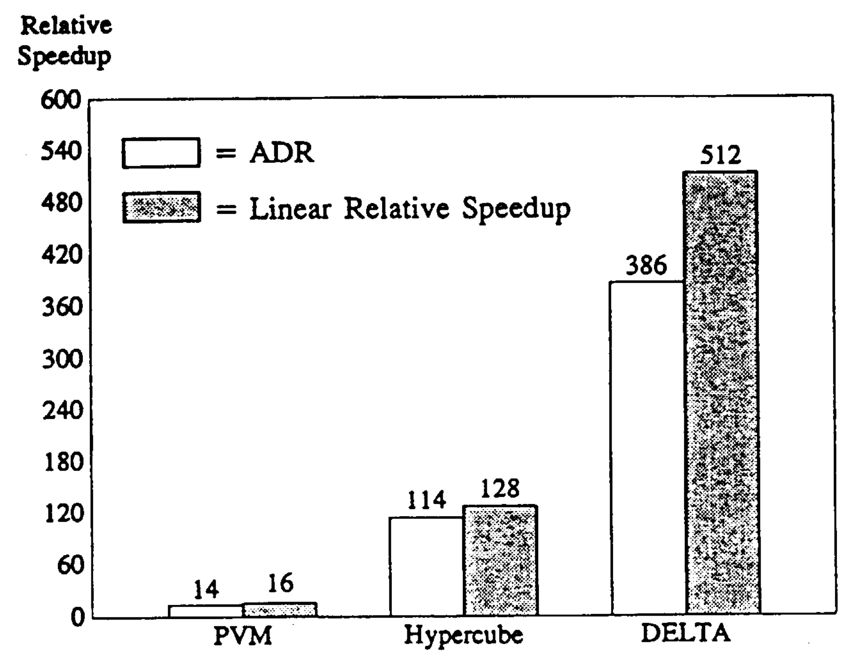
- Very efficient solution technique for nonlinear transient dynamic analysis

Parallel ADR Algorithm



Maximum Relative Speedups





Progress To–Date

- Completed development of quad. shell element for linear stress, buckling and vibration.
- Assessed alternative formulations.
- Completed development of compatible beam element for linear stress, buckling and vibration.
- Both have consistent and diagonal mass matrices, consistent loads, and element stress resultant recovery.
- Performed review and upgrade of nonlinear solution strategy.
- Applied elements to Grumman shear buckling panel with good results.

Future Plans and Summary

- Complete development of compatible 3–node triangle for linear stress, buckling and vibration.
- Derive the internal force vectors for geometrically nonlinear problems (beam, quad., and triangle).
- Validate the combined use of the elements.
- Validate nonlinear implementations.